Student's Name: ______________________

Student Number: | | | | | | | | | |

Teacher's Name: ______________________

ABBOTSLEIGH

2023
HIGHER SCHOOL CERTIFICATE
Assessment 4
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen.
- **NESA approved** calculators may be used.
- **NESA approved** reference sheet is provided.
- All necessary working should be shown in every question.
- Make sure your HSC candidate Number is on the front cover of each booklet.
- Start a new booklet for Each Question.
- Answer the Multiple Choice questions on the answer sheet provided.
- If you do not attempt a whole question, you must still hand in the Writing Booklet, with the words **'NOT ATTEMPTED'** written clearly on the front cover.

Total marks – 100

- Attempt Sections I and II.
- All questions are of equal value.

Section I — Pages 3 - 9

10 marks
- Attempt Questions 1–10.
- Allow about 15 minutes for this section.

Section II — Pages 9 - 19

90 marks
- Attempt Questions 11– 16.
- Allow about 2 hrs and 45 minutes for this section.

Outcomes to be assessed:

<u>Mathematics Extension 2</u>

HSC :
A student

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

MEX12-3 uses vectors to model and solve problems in two and three dimensions

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

MEX12-5 applies techniques of integration to structured and unstructured problems

MEX12-6 uses mechanics to model and solve practical problems

MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

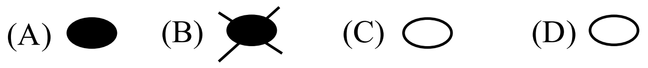
MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

SECTION I

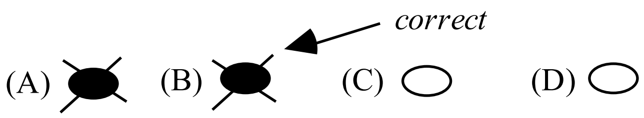
10 marks
Attempt Questions 1 – 10
Use the multiple-choice answer sheet
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.



If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows.



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1 Consider the following statements:

1. $z\overline{z} = |z|^2$
2. $\overline{z}^{\,-1} = \dfrac{z}{|z|^2}$

Which of the following is correct about the above statements:

A. Only statement 1 is true.

B. Only statement 2 is true.

C. Both statement 1 and 2 are true.

D. Neither statement 1 or 2 are true.

2 What is the modulus of $\dfrac{4+2i}{1-2i}$

A. $2\sqrt{5}$

B. 4

C. 3

D. 2

3 Ms Rennie states that if you cross the Pacific Highway using the overbridge, you won't get hurt. The contrapositive of this statement is:

A. If you don't cross the Pacific Highway using the overbridge, you will get hurt.

B. If you get hurt, then you didn't cross the Pacific Highway using the overbridge.

C. Crossing the Pacific Highway without using the overbridge means you will get hurt was not stated by Ms Rennie.

D. If you cross the Pacific Highway using the overbridge, you could get hurt.

4 The indefinite integral $\int x^3(x^4-1)^2\,dx$ is:

A. $\dfrac{1}{12}(x^4-1)^2+C$

B. $\dfrac{1}{4}(x^4-1)^3+C$

C. $x(x^4-1)+C$

D. $\dfrac{1}{12}(x^4-1)^3+C$

5 A particle moving in a straight line has its' velocity, v, given by $v = k(a - x)$, where a is a constant and x is the particle's displacement from the point O.

If the particle is initially at O, which of the following is an expression for x?

A. $x = a\left(1 + e^{-kt}\right)$

B. $x = a\left(1 + e^{kt}\right)$

C. $x = a\left(1 - e^{-kt}\right)$

D. $x = a\left(1 - e^{kt}\right)$

6 If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, then $\left|\hat{a} + \hat{b} + \hat{c}\right|$ is:

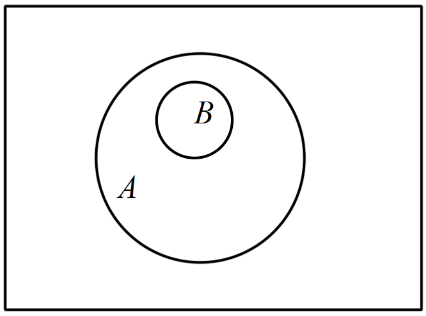
A. 1

B. $\sqrt{2}$

C. $\sqrt{3}$

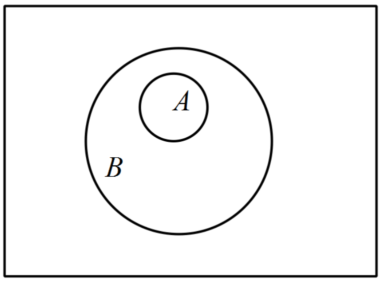
D. 2

7 The Venn diagram below shows $B \Rightarrow A$:

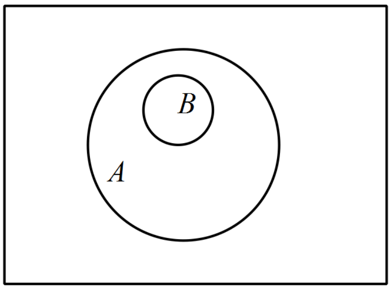


Which diagram below represents the $\neg A \Rightarrow \neg B$?

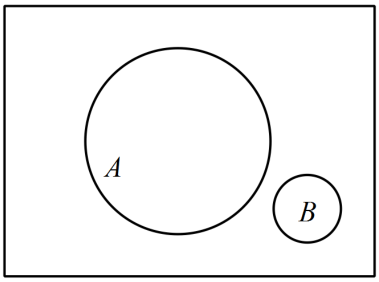
A.



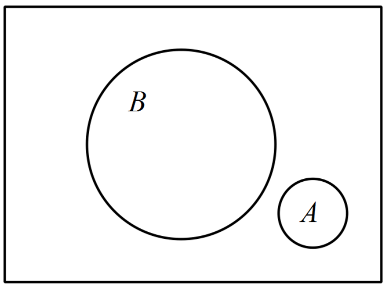
B.



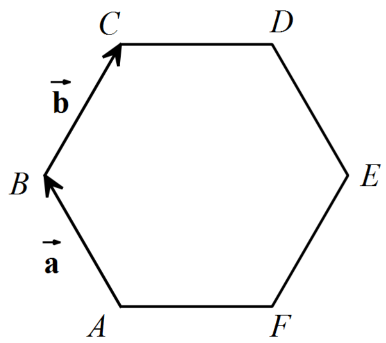
C.



D.



8 If $\vec{a}$ and $\vec{b}$ are the vectors forming consecutive sides of a regular hexagon *ABCDEF*, as shown, then the vector representing the side *CD* is:



A. $\vec{a}+\vec{b}$

B. $\vec{a}-\vec{b}$

C. $\vec{b}-\vec{a}$

D. $-(\vec{a}+\vec{b})$

9 Which of the following integrals evaluates to the largest value?

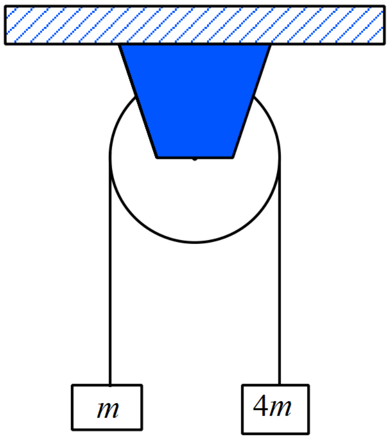
A. $\displaystyle\int_{-\pi}^{\pi} x\sin x \, dx$

B. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x^3)\ dx$

C. $\displaystyle\int_{-\pi}^{\pi} e^{-x^2}\, dx$

D. $\displaystyle\int_{-1}^{1} \tan^{-1} x^3\ dx$

10 Below is a diagram showing two objects with masses of m and $4m$ kg on either end of light inextensible strings that pass through a smooth pulley. Both objects are released from rest simultaneously.



NOT TO SCALE

Let g be the acceleration due to gravity. After 4 seconds, which of the following is true?

A. The heavier object has a speed of $\frac{3g}{5}$ ms^{-1}.

B. The heavier object has travelled $\frac{24g}{5}$ metres.

C. The heavier object has an acceleration of $\frac{3g}{4}$ ms^{-2}.

D. The heavier object has stopped moving as the lighter object has hit the pulley.

End of Section I

SECTION II

Total Marks – 90
Attempt Questions 11 - 16
All questions are of equal value

Answer each question in a **SEPARATE** writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) If $z = 3 + 2i$ and $w = 2 - i$, find, the following, writing your answer in the form $a + ib,\ a, b \in \mathbb{R}$:

(i) $z\,\overline{w}$ **2**

(ii) $\dfrac{z}{i\,w}$ **2**

(b) Find $\displaystyle\int \frac{x+34}{(x-6)(x+2)}\,dx$ **3**

(c) Prove that $\sqrt{3}$ is irrational. **2**

(d) Find $\displaystyle\int \frac{dx}{2x^2+3x+4}$ **3**

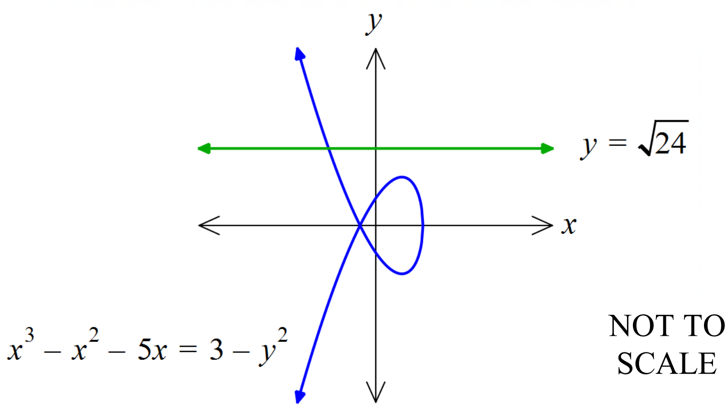
(e) Given $2+i$ is a zero of $P(x) = x^3 - x^2 - 7x + 15$, find the other zeroes. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given $\int_2^3 f(x)\,dx = \sqrt{7}$, find $\int_1^2 \frac{1}{x^2} f\left(1+\frac{2}{x}\right) dx$. **3**

(b) The equation $x^3 - x^2 - 5x = 3 - y^2$ implicitly defines the curve shown below.
The line $y = \sqrt{24}$ intersects this curve as shown.



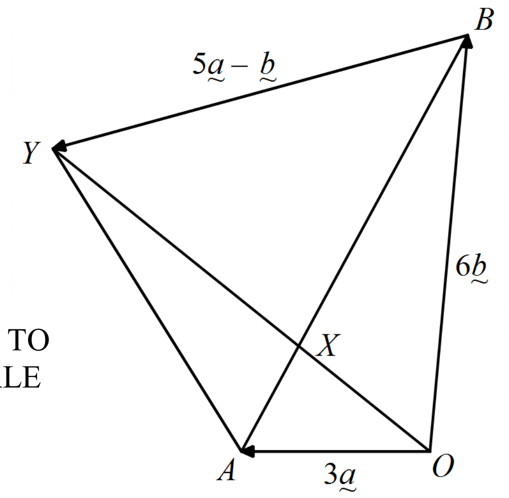
It can be shown that the equation $x^3 - x^2 - 5x + 21 = 0$ will determine the intersection between the line $y = \sqrt{24}$ and the implicitly defined curve.

(i) Explain, with reference to the graph above, why we know that there is only one real and two complex solutions to this cubic equation. **2**

(ii) Determine the two exact complex solutions to the equation $x^3 - x^2 - 5x + 21 = 0$. **2**

Question 12 continues on page 11

(c) The diagram below shows quadrilateral $OAYB$ with $\overrightarrow{OA} = 3\underset{\sim}{a}$ and $\overrightarrow{OB} = 6\underset{\sim}{b}$.



(i) Express $\overrightarrow{AB}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **1**

(ii) X is the point on AB such that $AX : XB = 1:2$ and $\overrightarrow{BY} = 5\underset{\sim}{a} - \underset{\sim}{b}$. **3**
Prove that $\overrightarrow{OX} = \frac{2}{5}\overrightarrow{OY}$.

(d) Let $a, b, c \in \mathbb{R}$.

(i) Prove the inequality:

$$a^2 + b^2 \geq 2ab$$ **1**

(ii) Hence or otherwise, prove:

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$ **1**

(iii) Hence show:

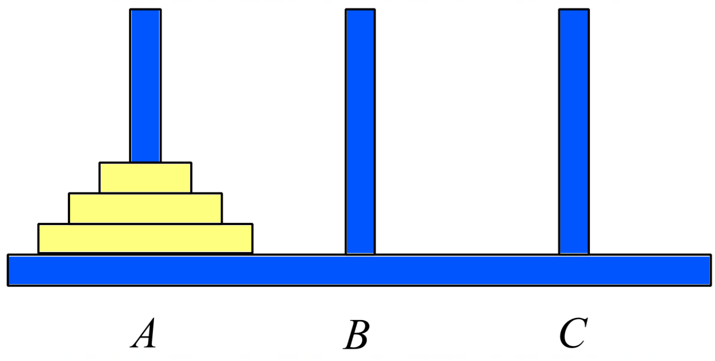
$$3(ab + bc + ca) \leq (a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$$ **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The Tower of Hanoi involves moving a set of circular disks from one peg, A, to another peg, C, one at a time, using a "helper tower", B, and no larger disk can ever be above a smaller disk. **3**

Prove by induction that the number of steps required to move n disks from one peg to another is $2^n - 1$.

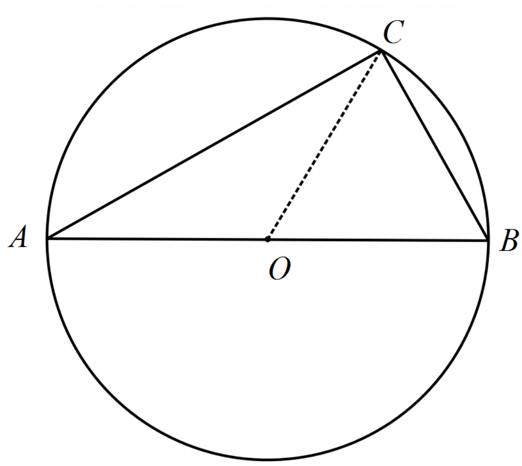


(b) The acceleration of a particle moving along the x-axis is given by $a = x(2-3x)$ ms^{-2}. Initially, $x = 0$ m, $v = 2\sqrt{2}$ ms^{-1}.

(i) Find an expression for v^2 as a function of x. **2**

(ii) Determine the values that x can take. **2**

(c) Using vectors, prove that the angle at the circumference of a semi-circle from the ends of the diameter is a right angle. **3**



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Question 13 continues on page 13

(d) A particle is moving along a straight line with simple harmonic motion, centre at *O*, and period $\frac{\pi}{3}$ seconds. **5**

When the particle is 0.48 m from *O*, its' speed is 2.16 ms^{-1}.

Calculate the total time in one complete oscillation that the particle has speed less than 2.88 ms^{-1}

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove by induction that $a^4 - 1$ is divisible by 16 for all odd integers, *a*. **3**

(b) Given $I_n = \int_0^{\sqrt{3}} \left(3 - x^2\right)^n \, dx.$

(i) Show that $I_n = \dfrac{6n}{2n+1} I_{n-1}, \quad n \geq 1.$ **3**

(ii) Hence evaluate I_3. **1**

(c) On the Argand diagram provided as an insert, sketch and label:

(i) $A = \left\{ z : \ z\overline{z} = 4, \ z \in C \right\}.$ **1**

(ii) $B = \left\{ z : |z| = \left| z - 2cis\dfrac{\pi}{4} \right|, \ z \in C \right\}$, labelling the axis intercepts. **2**

(iii) On the diagram, shade the region defined by: **1**

$$\left\{ z : z\overline{z} \leq 4, \ z \in C \right\} \cap \left\{ z : \text{Re}(z) + \text{Im}(z) \geq \sqrt{2}, \ \ z \in C \right\}.$$

(iv) Find the area of the shaded region in part (iii). **2**

(d) Prove that , for $x > 0$, $\ln x \leq x - 1$. **2**

End of Question 14

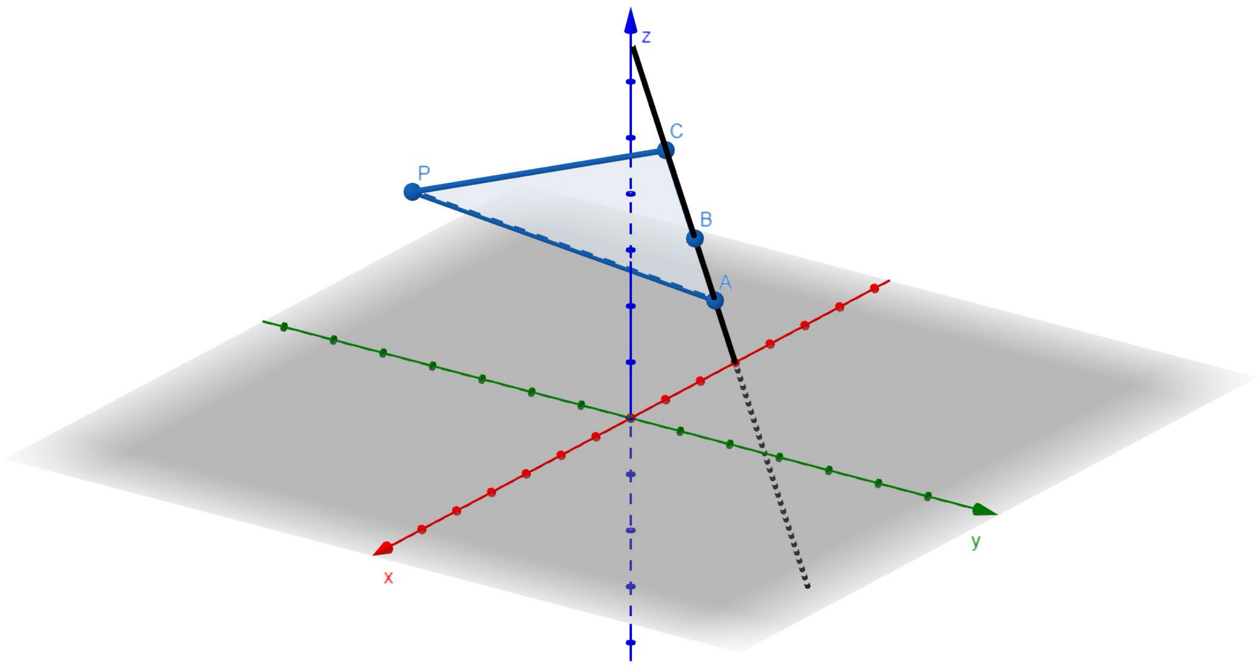
Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the line, l, with vector equation $\underset{\sim}{r}(\lambda) = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and the points

$A = (-1, 1, 2)$ and $B = (1, 2, 4)$ lie on the line. Let $\underset{\sim}{b} = \overrightarrow{AB}$, so

$$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ (DO NOT prove this).}$$

Let P be the point $(2, -3, 4)$.



(i) Find the projection of $\overrightarrow{AP}$ onto the line l, and hence the perpendicular distance from P to the line, l. **3**

(ii) Hence find the coordinates of the point, C on the line l such that the area of ΔAPC is 15 square units. **2**

Question 15 continues on page 16

(b) The infinite series C and S are defined by:

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \ldots$$

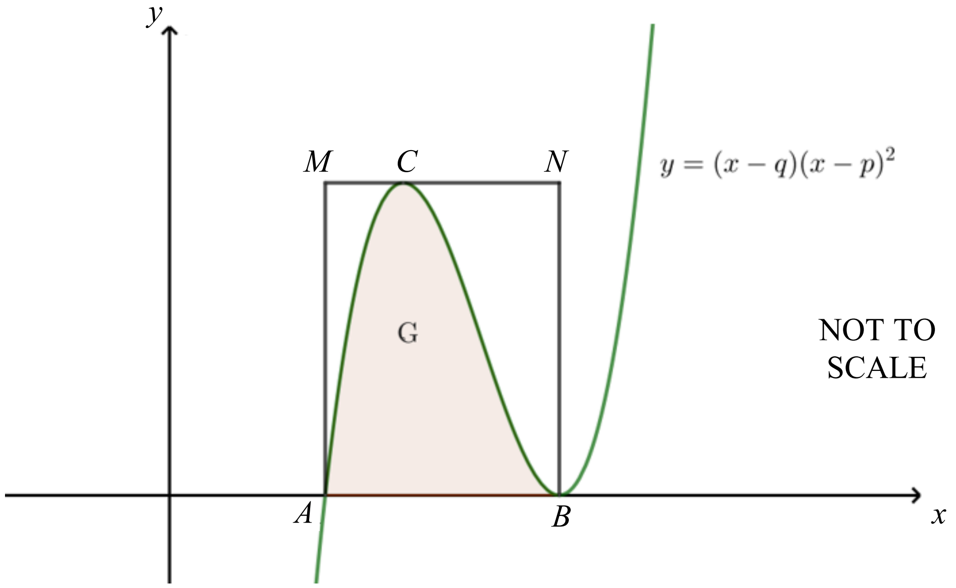
$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \ldots$$

(i) Show that $C + iS = \dfrac{2e^{i\theta}}{2 - e^{4i\theta}}$, given the two series are convergent. **2**

(ii) Hence show $S = \dfrac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$. **3**

Question 15 continues on page 17

(c) The diagram below shows the curve with equation $y = (x - q)(x - p)^2$, where p and q are positive constants.



The curve meets the x-axis at the points A and B.

The shaded region, G, is bounded by the curve and the x-axis.

(i) Show that the area of the shaded region is $\frac{1}{12}(p - q)^4$ **2**

(ii) The rectangle $AMNB$ passes through the local maximum at C. **3**

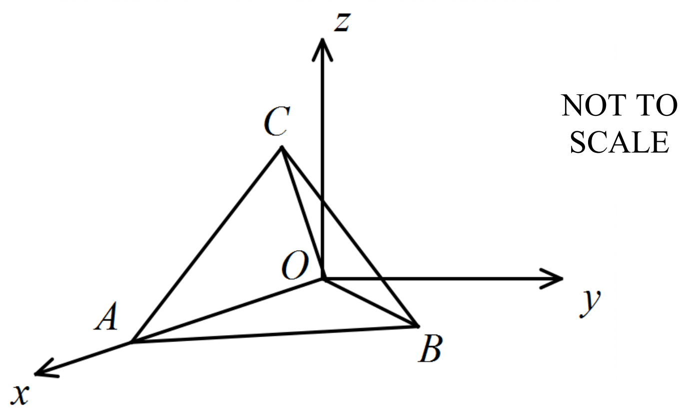
Show that the area of $AMNB$ is $\frac{16}{9}$ times as big as the region G, regardless of the values of p and q.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The faces of the tetrahedron $OABC$ comprise equilateral triangles of side length one unit. Its' base, OAB lies on the xy-plane. Two of the vertices are O and $A(1,0,0)$. The vertex C is above the xy-plane. **4**

Show that the coordinates of C are $\left(\frac{1}{2},\frac{\sqrt{3}}{6},\frac{\sqrt{6}}{3}\right)$.



(b) Let $a,b,c \in \mathbb{R}^+$.

(i) Prove the inequality: $\frac{a}{b}+\frac{b}{a} \geq 2$ **1**

(ii) Hence or otherwise prove: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$ **3**

Question 16 continues on page 18

Question 16 (continued). **Marks**

(c) An object of mass 1kg is dropped from a hot air balloon. The object is acted on by gravity and experiences a resistive force of kv Newtons, where k is a constant of proportionality.

(i) Taking down as the positive direction and using $F = mg - kv$ as the equation for the resultant force, show that the velocity, v, of the object t seconds after being dropped is given by: **2**

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

(ii) Two seconds after the first object is released, a second object of mass 1 kg is projected downward with an initial velocity of u ms^{-1}. It is also acted upon by gravity and a resistive force of kv Newtons. Show that the velocity, V, of this second object t seconds after the first object is released is given by: **2**

$$V = \frac{g}{k} - e^{-k(t-2)}\left(\frac{g - ku}{k}\right)$$

(iii) If the second object is projected at its' terminal velocity $\left(ie: u = \frac{g}{k}\right)$, find an expression for the distance fallen by the second object, y_2, t seconds after the first object is released. **2**

(iv) It can be shown that the vertical distance, y_1, travelled by the first object is given by: **1**

$$y_1 = \frac{g}{k}\left(t + \frac{1}{k}\left(e^{-kt} - 1\right)\right) \qquad \text{DO NOT prove this.}$$

Show that the time that the two objects will collide is given by:

$$t = -\frac{1}{k}\ln(1 - 2k).$$

End of paper

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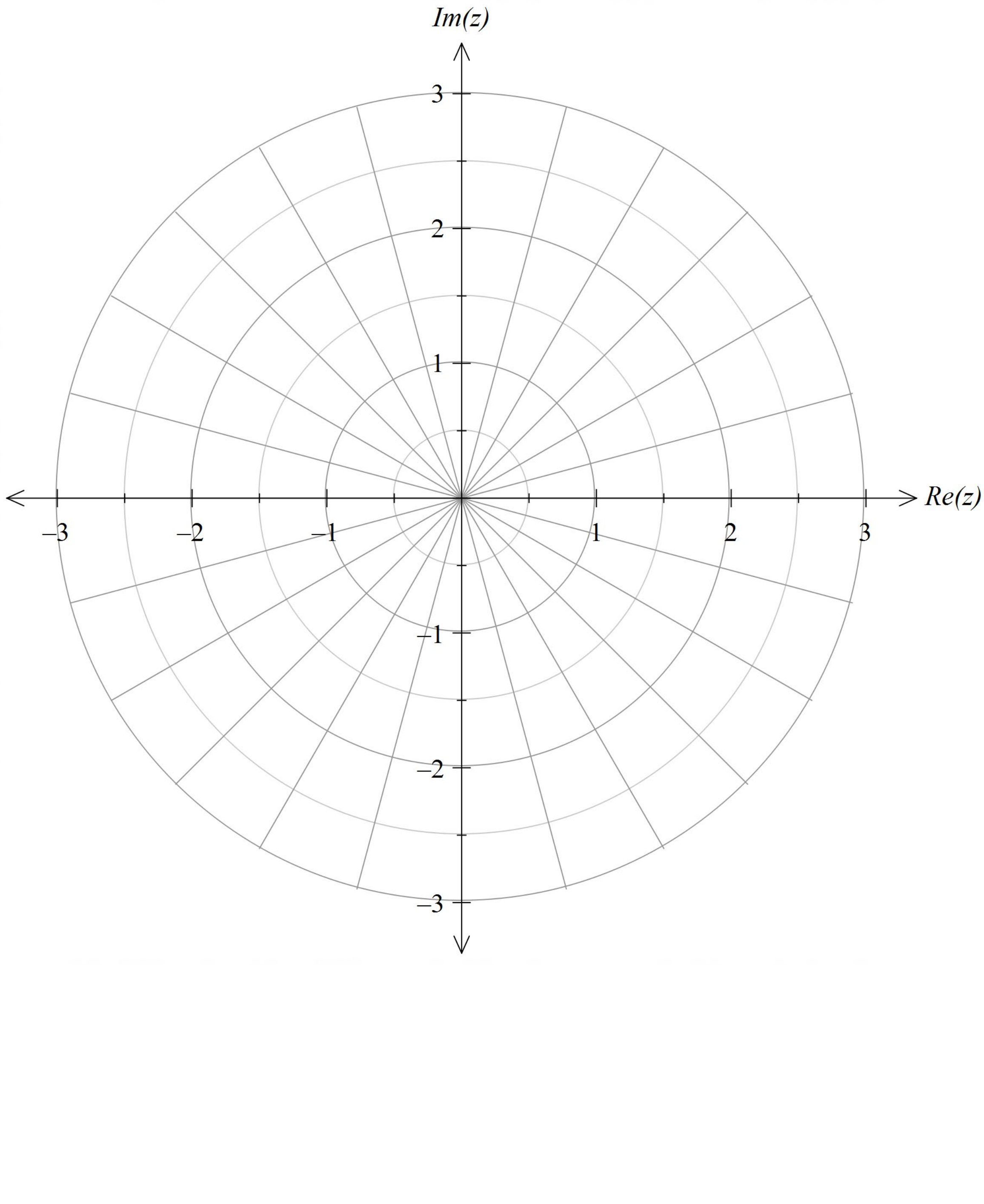
Argand diagram for Question 14(c).



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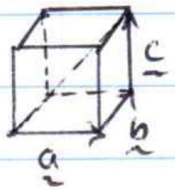
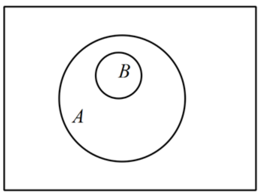
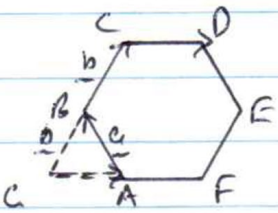
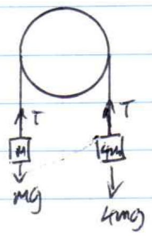
Extension 2 Task 4 2023 Solutions

Section I

1.	Let $z = x + iy$ $z\bar{z} = (x+iy)(x-iy)$ $= x^2 + y^2$ $= \lvert z\rvert^2$ $\bar{z}^{-1} = \frac{1}{x-iy} \times \frac{x+iy}{x+iy}$ $= \frac{x+iy}{x^2+y^2}$ $= \frac{z}{\lvert z\rvert^2}$ $\therefore$ both statements are true	**C**
2.	$\frac{4+2i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{4+8i+2i-4}{1+4}$ $= \frac{10i}{5}$ $= 2i$ $\therefore \left\lvert \frac{4+2i}{1-2i} \right\rvert = 2$	**D**
3.	$A \Rightarrow B$ for contrapositive: $\neg B \Rightarrow \neg A$	**B**
4.	$\int x^3(x^4-1)^2\,dx = \frac{1}{4}\int 4x^3(x^4-1)^2\,dx$ $= \frac{1}{4}\frac{(x^4-1)^3}{3} + C$ $= \frac{1}{12}(x^4-1)^3 + C$	**D**

5.	$v = k(a-x)$ $\frac{dx}{dt} = k(a-x)$ $\int_0^x \frac{dx}{a-x} = \int_0^t k\,dt$ $\left[-\ln\lvert a-x\rvert\right]_0^x = \left[kt\right]_0^t$ $-\ln\lvert a-x\rvert - -\ln\lvert a\rvert = kt$ $\ln\left\lvert\frac{a}{a-x}\right\rvert = +kt$ $\frac{a}{a-x} = e^{kt}$ $\frac{a-x}{a} = e^{-kt}$ $a-x = ae^{-kt}$ $x = a(1-e^{-kt})$	**C**
6.	6. $\hat{a} + \hat{b} + \hat{c}$ will be the diagonal of a 1 unit cube. $\lvert\hat{a}+\hat{b}\rvert = \sqrt{1^2+1^2}$ $= \sqrt{2}$ $\lvert\hat{a}+\hat{b}+\hat{c}\rvert = \sqrt{\sqrt{2}^2+1^2}$ $= \sqrt{3}$. 	**C**
7.	If B implies A then Diagram below represents the $\neg A \Rightarrow \neg B$? 	**B**
8.	$\vec{CD} = \vec{GA}$ now $\vec{GA} = b - a$ $\therefore \vec{CD} = b - a$ 	**C**

9.	$\int_{-\pi}^{\pi} x\sin x\,dx$ $x\sin x$ is even and positive for $-\pi \le x \le \pi$ $x\sin x$ attains a value of at least $\pi/2$ (when $x = \pi/2$) $\int_{-\pi/2}^{\pi/2} \cos(x^3)\,dx$ $\cos(x^3)$ is even and attain a value of 1 (at $x = 0$); however, it is not positive for the entire domain $-\pi/2 \le x \le \pi/2$ $\therefore < \int_{-\pi}^{\pi} x\sin x\,dx$. $\int_{-\pi}^{\pi} e^{-x^2}\,dx$ e^{-x^2} is an even function and positive for $-\pi \le x \le \pi$ e^{-x^2} attains a max value of 1 (at $x = 0$) $\therefore < \int_{-\pi}^{\pi} x\sin x\,dx$. $\int_{-1}^{1} \tan^{-1}(x^3)\,dx = 0$ as $\tan^{-1}(x^3)$ is odd.	**A**
10.	$F = 4mg - T - (mg - T)$ $= 3mg$.  $\therefore (m+4m)a = 3mg$. $a = \frac{3g}{5}$. $\therefore$ not C So $\frac{dv}{dt} = \frac{3g}{5}$ $v = \int_0^t \frac{3g}{5}\,dt$ $= \left[\frac{3gt}{5}\right]_0^t$ $= \frac{3gt}{5}$. when $t = 4$. $v = \frac{3g \times 4}{5}$ $= \frac{12g}{5}$. $\therefore$ not A. $v = \frac{3gt}{5}$ $\frac{dx}{dt} = \frac{3gt}{5}$. $x = \int_0^t \frac{3gt}{5}\,dt$ $= \left[\frac{3gt^2}{10}\right]_0^t$ $= \frac{3gt^2}{10}$ at $t = 4$ $x = \frac{3g}{10} \times 4^2$ $= \frac{48g}{10}$ $= \frac{24g}{5}$	**B**

Section II

Question 11 (15 marks) **Marks**

(a)	(i)	$z\,\overline{w} = (3+2i)(2+i)$ ☑ $= 6+3i+4i+2i^2$ $= 4+7i$ ☑	**2**
	(ii)	$\dfrac{z}{i\,w} = \dfrac{(3+2i)}{i(2+i)}$ $= \dfrac{3+2i}{2i-i^2}$ $= \dfrac{3+2i}{1+2i}$ $= \dfrac{3+2i}{1+2i} \times \dfrac{1-2i}{1-2i}$ ☑ $= \dfrac{3-6i+2i-4i^2}{1-4i^2}$ $= \dfrac{7-4i}{5}$ *or* $\dfrac{7}{5} - \dfrac{4i}{5}$ ☑	**2**
(b)	$\int \dfrac{x+34}{(x-6)(x+2)}\,dx = \int \dfrac{A}{x-6} + \dfrac{B}{x+2}\,dx$ *ie*: $A(x+2)+B(x-6) = x+34$ *When* $x=6$ $\quad 8A+0=40$ $\therefore A=5$ *When* $x=-2$ $\quad 0-8B=32$ $B=-4$ ☑ $\therefore \int \dfrac{x+34}{(x-6)(x+2)}\,dx = \int \dfrac{5}{x-6} + \dfrac{-4}{x+2}\,dx$ $= 5\ln\lvert x-6\rvert - 4\ln\lvert x+2\rvert + C$ ☑ $= \ln\lvert x-6\rvert^5 - 4\ln\lvert x+2\rvert + C$ $= \ln\dfrac{\lvert x-6\rvert^5}{\lvert x+2\rvert} + C$ ☑		**3**

(c)	Prove that $\sqrt{3}$ is irrational. Proof by contradiction Assume $\sqrt{3}$ is rational $ie: \sqrt{3} = \frac{p}{q}$ *where &are positive integers with no common factors other than* 1 ✓ *So* $\sqrt{3}q = p$ $3q^2 = p^2$ This implies that p^2 has a factor of 3. But square numbers have pairs of each factor by definition $\therefore$ p has a factor of 3 ie: $p = 3m$ *So* $3q^2 = (3m)^2$ $q^2 = 3m^2$ It follows that q^2 has a factor of 3 and so q has a factor of 3 *So* Both p and q have a factor of 3 which contradicts the assumption $\therefore \sqrt{3}$ is irratonal by contradiction ✓	**2**
(d)	$\int \frac{dx}{2x^2+3x+4} = \frac{1}{2}\int \frac{1}{x^2+\frac{3}{2}x+2}\,dx$ $= \frac{1}{2}\int \frac{1}{x^2+\frac{3}{2}x+\frac{9}{16}+2-\frac{9}{16}}\,dx$ ✓ $= \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\,dx$ ✓ $= \frac{1}{2}\times\left(\frac{4}{\sqrt{23}}\tan^{-1}\left(\frac{\left(x+\frac{3}{4}\right)}{\frac{\sqrt{23}}{4}}\right)\right)+C$ $= \frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{4x+3}{\sqrt{23}}\right)+C$ ✓	**3**

(e)	$P(x) = x^3 - x^2 - 7x + 15$ Since the coefficients are rational, any complex roots occur in conjugate pairs (conjugate root theorem) $\therefore$ if $\alpha = 2+i$ is a root then $\beta = 2-i$ is also a root. ☑ Now $\alpha+\beta = 2+i+2-i$ $= 4$ $\alpha\beta = (2+i)(2-i)$ $= 5$ $\therefore$ The quadratic with roots α & β is $x^2 - 4x + 5 = 0$ $\therefore x^2-4x+5$ is a factor of $P(x)$ So $P(x) = (x^2-4x+5)(x+3)$ By inspection ☑ So roots: $2+i$, $2-i$, -3.	OR Sum of the roots $\alpha+\beta+\gamma = +1$ $(2+i)(2-i)+\gamma = 1$ $4+\gamma = 1$ $\gamma = -3$	3

Question 12 (15 marks) **Marks**

(a)

Given $\int_2^3 f(x)\,dx = \sqrt{7}$

$I = \int_1^2 \frac{1}{x^2} f\left(1+\frac{2}{x}\right) dx.$

Let $u = 1+\frac{2}{x}$

$du = -\frac{2}{x^2}\,dx.$

when $x=1$ $u = 1+\frac{2}{1} = 3$

when $x=2$ $u = 1+\frac{2}{2} = 2.$ ☑

$\therefore I = -\frac{1}{2}\int_1^2 -\frac{2}{x^2} f\left(1+\frac{2}{x}\right) dx$

$= -\frac{1}{2}\int_3^2 f(u)\,du.$ ☑

$= \frac{1}{2}\int_2^3 f(u)\,du$

$= \frac{1}{2}(\sqrt{7})$ as u is dummy variable

$= \sqrt{7}/2.$ ☑

OR

$\int_2^3 f(x)\,dx = \sqrt{7}$

$\therefore [F(x)]_2^3 = \sqrt{7}$

$F(3)-F(2) = \sqrt{7}$

$I = \int_1^2 \frac{1}{\lambda^2} f\left(1+\frac{2}{\lambda}\right) dx$

$= -\frac{1}{2}\int_1^2 -\frac{2}{\lambda^2} f\left(1+\frac{2}{\lambda}\right) dx$

$= -\frac{1}{2}\left[F\left(1+\frac{2}{\lambda}\right)\right]_1^2$ as $\frac{d}{dx}\left(\frac{2}{x}\right) = \frac{-2}{x^2}$

$= -\frac{1}{2}[F(2)-F(3)]$

$= \frac{1}{2}[F(3)-F(2)]$

$= \frac{\sqrt{7}}{2}$

3

(b) (i)

(i) Solving $x^3 - x^2 - 5x = 3 - y^2$

& $y = \sqrt{24}$

simultaneously yields

$x^3 - x^2 - 5x + 21 = 0$

This is a cubic so will have 3 roots. ☑

From the graph, there is only one point of intersection

$\Rightarrow$ 1 real root, 2 complex roots. ☑

2

	(ii)	Sub $x=-3$ (from graph $x<0$ & from eqn x is odd) $P(x)=x^3-x^2-5x+21$ $P(-3)=(-3)^3-(-3)^2-5(-3)+21$ $=-27-9+15+21$ $=0$ $\therefore x+3$ is a root of $P(x)$ ✓ i.e. $P(x)=(x+3)(x^2-4x+7)$ by inspection The complex roots will be the solution to: $x^2-4x+7=0$ $x^2-4x+4=-7+4$ $(x-2)^2=-3$ $x-2=\pm\sqrt{3}i$ $x=2\pm\sqrt{3}i$ $\therefore$ complex roots are $x=2+\sqrt{3}i,\ 2-\sqrt{3}i$ ✓ → OR Let roots be α,β & -3 where $\alpha=x+iy,\ \beta=x-iy$ sum of roots $x+iy+x-iy+-3=1$ $2x=4$ $x=2$ Product of the roots $-3(x+iy)(x-iy)=-21$ $x^2+y^2=7$ $(2)^2+y^2=7$ $y^2=3$ $y=\pm3$	**2**
(c)	(i)	$\vec{AB}=\vec{OB}-\vec{OA}$ $=6\underset{\sim}{b}-3\underset{\sim}{a}$ $=3(2\underset{\sim}{b}-\underset{\sim}{a})$ ✓	**1**
	(ii)	$\vec{OY}=\vec{OB}+\vec{BY}$ $=6\underset{\sim}{b}+5\underset{\sim}{a}-\underset{\sim}{b}$ $=5(\underset{\sim}{a}+\underset{\sim}{b})$ ✓ Since $AX:XB=1:2$ $\vec{AX}=\frac{1}{3}\vec{AB}$ Now $\vec{OX}=\vec{OA}+\vec{AX}$ $=3\underset{\sim}{a}+\frac{1}{3}(6\underset{\sim}{b}-3\underset{\sim}{a})$ $=3\underset{\sim}{a}+2\underset{\sim}{b}-\underset{\sim}{a}$ $=2(\underset{\sim}{a}+\underset{\sim}{b})$ ✓ $\therefore \vec{OX}:\vec{OY}=2(\underset{\sim}{a}+\underset{\sim}{b}):5(\underset{\sim}{a}+\underset{\sim}{b})$ $=2:5$ So $\vec{OX}=\frac{2}{5}\vec{OY}$ ✓	**3**

(d)	(i)	We know, for $a, b \in \mathbb{R}$ $(a-b)^2 \geq 0$ $\therefore a^2+b^2-2ab \geq 0$ $a^2+b^2 \geq 2ab$ ① ☑	**1**
	(ii)	Similarly $b^2+c^2 \geq 2bc$ ② & $c^2+a^2 \geq 2ac$ ③ Summing ①, ② & ③: $a^2+b^2+b^2+c^2+c^2+a^2 \geq 2ab+2bc+2ca$ $2a^2+2b^2+2c^2 \geq 2(ab+bc+ca)$ $a^2+b^2+c^2 \geq ab+bc+ca$. ☑	**1**
	(iii)	(iii) $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$ $\geq ab+bc+ca+2(ab+bc+ca)$ from (ii) $= 3(ab+bc+ca)$ ☑ also $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$ $\leq a^2+b^2+c^2+2(a^2+b^2+c^2)$ $=$ from (ii) $= 3(a^2+b^2+c^2)$ $\therefore 3(ab+bc+ca) \leq (a+b+c)^2 \leq 3(a^2+b^2+c^2)$ ☑	**2**

Question 13 (15 marks) **Marks**

(a)

Step 1: Let $n=1$
Since there are no other disks it takes 1 move to transfer the disk from peg A to peg C
$n=1$ moves $= 2^1 - 1$
$= 1$
$\therefore$ true for $n=1$ ☑

Step 2: Assume true for $n=k$
So it takes $2^k - 1$ moves to transfer the disks from peg A to peg C

Step 3: Prove true for $n=k+1$
using the assumption in step 2. It will take $2^k - 1$ moves to transfer the top k disks from peg A to peg B
It will then take 1 move to transfer the $(k+1)^{th}$ (last) disk from peg A to C
It then takes $2^k - 1$ moves to transfer the k disks from peg B to peg C ☑

$\therefore$ moves $= (2^k - 1) + 1 + (2^k - 1)$
$= 2.2^k - 1$
$= 2^{k+1} - 1$
as required. ☑

Step 4: The result holds by the inductive process.

3

(b) (i)

We have:

$$v\frac{dv}{dx} = 2x - 3x^2$$

$$\int_{2\sqrt{2}}^{v} v\frac{dv}{dx}\,dx = \int_0^x (2x - 3x^2)\,dx$$

$$\left[\tfrac{1}{2}v^2\right]_{2\sqrt{2}}^{v} = \left[x^2 - x^3\right]_0^x$$ ☑

OR

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - 3x^2$$

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}(2\sqrt{2})^2 = x^2 - x^3$$

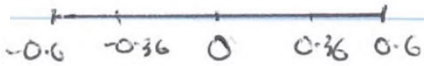
$$v^2 - 8 = 2x^2 - 2x^3$$

$$v^2 = 8 + 2x^2 - 2x^3$$ ☑

2

			Marks
	(ii)	Hence $v = \pm\sqrt{8+2x^2-2x^3}$ but, when $x=0$, $v=2\sqrt{2}$ $\therefore\ v = \sqrt{8+2x^2-2x^3}$ For v to exist: $8+2x^2-2x^3 \geq 0$ or $x^3-x^2-4 \leq 0$ ✓ let $x=2$: $2^3-2^2-4=0$. $\therefore$ $(x-2)$ is a factor of x^3-x^2-4 So $(x-2)(x^2+x+2) \leq 0$ by inspection So $x-2=0$ or $x^2+x+2=0$ $x=2$ $\quad \Delta = 1^2-4(1)(2) < 0$ $\therefore$ $x=2$ is only real solution $\therefore$ $x \leq 2$ (as $x=0$ is initial condition) ✓	2
(c)		Since O is the centre of the circle $\lvert\vec{OA}\rvert = \lvert\vec{OB}\rvert = \lvert\vec{OC}\rvert$ radii of circle. also $\vec{OA} = -\vec{OB}$ Now $\vec{AC} = \vec{OC} - \vec{OA}$ $= \vec{OC} + \vec{OB}$ $\quad (\vec{OA} = -\vec{OB})$ & $\vec{CB} = \vec{OB} - \vec{OC}$ ✓ If $\angle ACB = 90°$ then $\vec{AC} \cdot \vec{CB} = 0$ ✓ $\vec{AC} \cdot \vec{CB} = (\vec{OC}+\vec{OB})(\vec{OB}-\vec{OC})$ $= \vec{OC}\cdot\vec{OB} - \vec{OC}\cdot\vec{OC} + \vec{OB}\cdot\vec{OB} - \vec{OC}\cdot\vec{OB}$ $= -\lvert\vec{OC}\rvert^2 + \lvert\vec{OB}\rvert^2$ $= 0$ as $\underset{\sim}{a}\cdot\underset{\sim}{a} = \lvert\underset{\sim}{a}\rvert^2$ & $\lvert\vec{OB}\rvert = \lvert\vec{OC}\rvert$ ✓ $\therefore$ $\angle ACB = 90°$	3

(d)



Since the particle is moving with SHM, we have.

$v^2 = n^2(a^2 - x^2)$

now, period $= \frac{2\pi}{n}$

$\therefore \frac{2\pi}{n} = \frac{\pi}{3}$.

$n = 6$ ✓

so $v^2 = 36(a^2 - x^2)$

When $x = 0.48$, $v = \pm 2.16$

$\therefore (2.16)^2 = 36(a^2 - (0.48)^2)$

$0.1296 = a^2 - 0.2304$

$a^2 = 0.36$

$a = \pm 0.6$ ✓

Using $x = a\sin(nt)$ for simple harmonic motion about O:

$x = 0.6 \sin 6t$

now, when $v = 2.88$

$(2.88)^2 = 36((0.6)^2 - x^2)$

$0.2304 = 0.36 - x^2$

$x^2 = 0.1296$

$x = \pm 0.36$

In SHM, speed is greatest in the mean position, therefore we need to find when the particle is between

$-0.6 \le x \le 0.36 \quad and \quad 0.36 \le x \le 0.6$ ✓

When $x = 0.36$

$0.36 = 0.6 \sin 6t$

$\sin 6t = 0.6$

$6t = 0.6435$

$t = 0.1073$ ✓

Since period is $\frac{\pi}{3}$, time to get to $x = 0.6$

$is \quad t = \frac{\pi}{3} \div 4$

$= 0.2618$ sec

$\therefore t \text{ when } v < 2.88 = 4 \times (0.2618 - 0.1073)$

$= 4 \times 0.1545$

$= 0.6182$ sec ✓

5

Question 14 (15 marks)

Marks

(a)

Prove $a^4 - 1$ is divisible by 16 for any odd 'a'

Step 1: Let $a = 1$

$1^4 - 1 = 0$

which is true (but trivial)

note if $a = 3$

$3^4 - 1 = 81 - 1$

$= 80$

$= 16 \times 5$ ✓

$\therefore$ true for $a = 3$ as well.

Step 2: assume true for some odd number, k.

ie $k^4 - 1 = 16M$ for some $M \in \mathbb{Z}$

Step 3: Prove true for the next odd number, $a = k+2$

i.e. $(k+2)^4 - 1 = 16N$, $N \in \mathbb{Z}$

$LHS = k^4 + 4(2)k^3 + 6(2)^2k^2 + 4(2)^3k + 2^4 - 1$

$= k^4 + 8k^3 + 24k^2 + 32k + 16 - 1$

$= 16M + 8k^3 + 24k^2 + 32k + 16$ from assumption

$= 16(M + 2k + 1) + 8k^2(k+3)$ ✓

Now, since k is odd, $k+3$ is even. $\therefore k + 3 = 2P$, $P \in \mathbb{Z}$

$= 16(M + 2k + 1) + 8k^2(2P)$

$= 16(M + 2k + 1 + k^2P)$

$= 16N$ as all terms in brackets are integers ✓

Step 4: The result holds by the inductive process.

3

OR

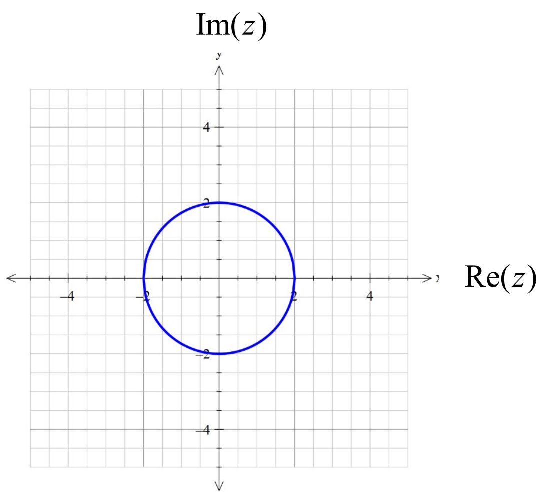
Step 2: Assume true for some odd number $2k+1$

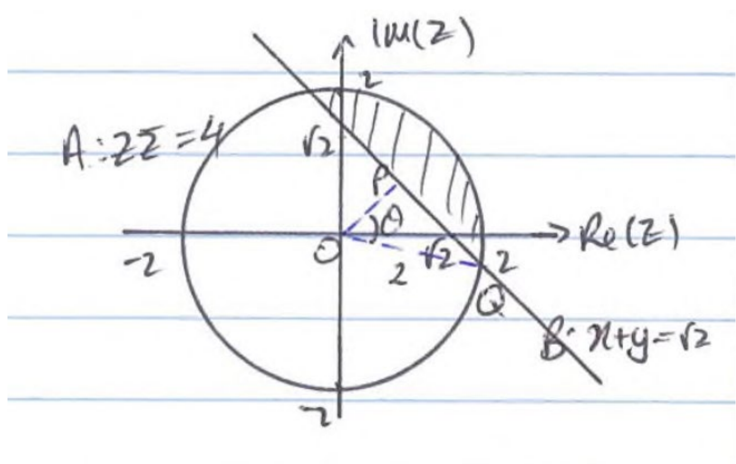
ie: $(2k+1)^2 - 1 = 16M \quad where\ M \in \mathbb{Z}$

Step 3: Prove true for next odd number $(2k+1)+2$

$ie:\ Show\ [(2k+1)+2]^4 - 1 = 16Q \quad where\ Q \in \mathbb{Z}$

$$\begin{aligned}
LHS &= (2k+1)^4 + 4(2k+1)^3(2) + 6(2k+1)^2(2)^2 + 4(2k+1)(2)^3 + (2)^4 - 1 \\
&= (2k+1)^4 - 1 + 4(2k+1)^3(2) + 6(2k+1)^2(2)^2 + 4(2k+1)(2)^3 + (2)^4 \\
&= 16M + 4(2k+1)^3(2) + 6(2k+1)^2(2)^2 + 4(2k+1)(2)^3 + (2)^4 \qquad by\ assumption \\
&= 16M + 64k^3 + 192k^2 + 208K + 80 \\
&= 16(M + 4k^3 + 12k^2 + 13k + 5) \\
&= 16Q \qquad where\ Q \in \mathbb{Z}
\end{aligned}$$

(b)	(i)	$I_n = \int_0^{\sqrt{3}} (3-x^2)^n \, dx$ (i) Integrating by parts with $u = (3-x^2)^n \qquad v' = 1$ $u' = -2nx(3-x^2)^{n-1} \qquad v = x$ ✓ $\therefore I_n = \left[x(3-x^2)^n\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} -2nx^2(3-x^2)^{n-1} \, dx$ $= 0 + 2n\int_0^{\sqrt{3}} \left[x^2(3-x^2)^{n-1} + 3(3-x^2)^{n-1} - 3(3-x^2)^{n-1}\right] dx$ $= -2n\int_0^{\sqrt{3}} \left[(3-x^2)^{n-1}(3-x^2) - 3(3-x^2)^{n-1}\right] dx$ ✓ $= -2n\int_0^{\sqrt{3}} (3-x^2)^n \, dx + 6n\int_0^{\sqrt{3}} (3-x^2)^{n-1} \, dx$ $= -2nI_n + 6nI_{n-1}$ $I_n(2n+1) = 6nI_{n-1}$ ✓ $I_n = \frac{6n}{2n+1} I_{n-1}$ as required.	**3**
	(ii)	$I_3 = \frac{6\times 3}{2(3)+1} \cdot I_2$ $= \frac{18}{7} \cdot \frac{12}{5} I_1$ $= \frac{18}{7} \times \frac{12}{5} \times \frac{6}{3} \int_0^{\sqrt{3}} dx$ $= \frac{432}{35}(\sqrt{3} - 0)$ $= \frac{432\sqrt{3}}{35}$ ✓	**1**
(c)	(i)	Let $z = x + iy$ Then $z\bar{z} = (x+iy)(x-iy)$ $= x^2 + y^2$ So $z\bar{z} = 4 \Rightarrow x^2 + y^2 = 4$ This is a circle, centre $(0,0)$, radius 2 ✓ 	**1**

	(ii)	(ii) $\|z\| = \|z - 2\text{cis}\,\pi/4\|$ now $2\text{cis}\,\pi/4 = 2\cos\pi/4 + 2i\sin\pi/4$ $= \sqrt{2}(1+i)$ then $\|z\|^2 = \|z - \sqrt{2}(1+i)\|^2$ ☑ $x^2+y^2 = (x-\sqrt{2})^2 + (y-\sqrt{2})^2$ $x^2+y^2 = x^2 - 2\sqrt{2}x + 2 + y^2 - 2\sqrt{2}y + 2$ $2\sqrt{2}(x+y) = 4$ $x+y = \frac{2}{\sqrt{2}}$ $x+y = \sqrt{2}$ x & y intercept is $\sqrt{2}$. ☑	**2**
	(iii)	 ☑	**1**
	(iv)	(iv) Distance OP on graph will be ½ the distance from O to $2\text{cis}\,\pi/4$ as the line is the perpendicular bisector of O & $2\text{cis}\,\pi/4$ $\|2\text{cis}\,\pi/4\| = \sqrt{2+2}$ $= 2.$ ☑ $\therefore OP = 1$ also $OQ = 2$ (radius of circle) $\therefore \theta = \cos^{-1}(\frac{1}{2})$ $= \pi/3$ So Area of sector in circle of radius 2 subtended by an angle of $2(\frac{\pi}{3})$ will be required area: $A = \frac{1}{2}2^2(\frac{2\pi}{3} - \sin\pi/3)$ $= \frac{4\pi}{3} - \sqrt{3}$ unit². ☑	**2**

(d)	d) Prove $\ln x \leq x - 1$ for $x > 0$ consider $f(x) = x - 1 - \ln x$. R.T.P. $f(x) \geq 0$ for $x > 0$ $f'(x) = 1 - \frac{1}{x}$. $f'(x) = 0 \Rightarrow$ max/min values $1 - \frac{1}{x} = 0$ $x = 1$ ☑ $f''(x) = \frac{1}{x^2} \Rightarrow f(x)$ concave up for all x so $x=1$ is absolute min of $f(x)$ $f(1) = 1 - 1 - \ln 1$ $= 0$ $\therefore f(x) \geq 0$, $x > 0$ ☑ as required.	2

Question 15 (15 marks) **Marks**

(a)	(i)	$r(\lambda) = \begin{bmatrix}-1\\1\\2\end{bmatrix} + \lambda\begin{bmatrix}2\\1\\2\end{bmatrix}$ $A=(-1,1,2)$ $B=(1,2,4)$ $\overrightarrow{AB} = \begin{bmatrix}2\\1\\2\end{bmatrix}$ $P=(2,-3,4)$ (i) $\overrightarrow{AP} = \begin{bmatrix}2--1\\-3-1\\4-2\end{bmatrix}$ $= \begin{bmatrix}3\\-4\\2\end{bmatrix}$ ✓ $\text{Proj}_{\underset{\sim}{b}} \overrightarrow{AP} = \dfrac{\underset{\sim}{b}\cdot\overrightarrow{AP}}{\lvert\underset{\sim}{b}\rvert^2}\underset{\sim}{b}$ $= \dfrac{2\times3-1\times4+2\times2}{2^2+1^2+2^2}\begin{bmatrix}2\\1\\2\end{bmatrix}$ $= \dfrac{6}{9}\begin{bmatrix}2\\1\\2\end{bmatrix}$ $= \dfrac{2}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}$ ✓ let d be perpendicular distance from P to l $\therefore d = \lvert \text{Proj}_{\underset{\sim}{b}} \overrightarrow{AP} - \overrightarrow{AP} \rvert$ $= \left\lvert \begin{bmatrix}\frac{4}{3}-3\\\frac{2}{3}+4\\\frac{4}{3}-2\end{bmatrix} \right\rvert$ $= \sqrt{(-5/3)^2+(14/3)^2+(-2/3)^2}$ $= \sqrt{\dfrac{225}{9}}$ $=$ 5 units ✓	**3**
	(ii)	Let C be some point on the line l: $C = \begin{bmatrix}-1\\1\\2\end{bmatrix} + \mu\begin{bmatrix}2\\1\\2\end{bmatrix}$ for some μ. ie. $C = (-1+2\mu,\ 1+\mu,\ 2+2\mu)$ $\therefore \overrightarrow{AC} = \begin{bmatrix}2\mu\\\mu\\2\mu\end{bmatrix}$ If the area $\triangle ACP = 15\ u^2$ we have $\frac{1}{2}\times\lvert\overrightarrow{AC}\rvert d = 15.$ $\therefore \frac{1}{2}\times\lvert\overrightarrow{AC}\rvert\times5 = 15$ $\lvert\overrightarrow{AC}\rvert = 6$ ✓	**2**

		So $\sqrt{(2\mu)^2+(\mu)^2+(2\mu)^2}=6$ $9\mu^2=36$ $\mu^2=4$ $\therefore \mu=\pm 2$ So $C=(-1+4,\ 1+2,\ 2+4)$ if $\mu=2$ $=(3,3,6)$ or $C=(-1-4,\ 1-2,\ 2-4)$ if $\mu=-2$ $=(-5,-1,-2)$ ✓	
(b)	(i)	$C=\cos\theta+\frac{1}{2}\cos5\theta+\frac{1}{4}\cos9\theta+\ldots$ $S=\sin\theta+\frac{1}{2}\sin5\theta+\frac{1}{4}\sin9\theta+\ldots$ i) $C+iS$ $=(\cos\theta+i\sin\theta)+\frac{1}{2}(\cos5\theta+i\sin5\theta)+\frac{1}{4}(\cos9\theta+i\sin9\theta)+\ldots$ $=\text{cis}\theta+\frac{1}{2}\text{cis}5\theta+\frac{1}{4}\text{cis}9\theta+\ldots$ $=\text{cis}\theta+\frac{1}{2}\text{cis}\theta\,\text{cis}4\theta+\frac{1}{4}\text{cis}\theta(\text{cis}4\theta)^2+\ldots$ This is a G.P with $a=\text{cis}\theta$ $r=\frac{1}{2}\text{cis}4\theta$ ✓ So $C+iS$ will be the limiting sum of this series. $C+iS=\dfrac{\text{cis}\theta}{1-\frac{1}{2}\text{cis}4\theta}$ $=\dfrac{e^{i\theta}}{1-\frac{1}{2}e^{4i\theta}}$ $=\dfrac{e^{i\theta}}{\frac{2-e^{4i\theta}}{2}}$ $=\dfrac{2e^{i\theta}}{2-e^{4i\theta}}$ ✓ as required.	2
	(ii)	To find S we need to realise the denominator of the expression in part (i) $C+iS=\dfrac{2e^{i\theta}}{2-e^{4i\theta}}\times\dfrac{2-e^{-4i\theta}}{2-e^{-4i\theta}}$ $=\dfrac{4e^{i\theta}-2e^{-3i\theta}}{4-2e^{-4i\theta}-2e^{4i\theta}+e^{0}}$ ✓ $=\dfrac{4e^{i\theta}-2e^{-3i\theta}}{4-2(e^{4i\theta}+e^{-4i\theta})+1}$ now $e^{4i\theta}+e^{-4i\theta}$ $=\text{cis}4\theta+\text{cis}(-4\theta)$ $=\cos4\theta+i\sin4\theta+\cos(-4\theta)+i\sin(-4\theta)$ $=2\cos4\theta$ as $\sin(-4\theta)=-\sin4\theta$ ✓ $\therefore C+iS=\dfrac{4e^{i\theta}-2e^{-3i\theta}}{5-4\cos4\theta}$ $S=\text{Im}(C+iS)$ $=\dfrac{4\sin\theta-2\sin(-3\theta)}{5-4\cos4\theta}$ $=\dfrac{4\sin\theta+2\sin(3\theta)}{5-4\cos4\theta}$ ✓ as required.	

<table>
<tr><td>(c)</td><td>(i)</td><td>

$A_a = \int_q^p (x-q)(x-p)^2\,dx$

Integrate by parts:

let $u = x-q$ $v' = (x-p)^2$

$u' = 1$ $v = \frac{1}{3}(x-p)^3$ ✓

$\therefore A_a = \left[\frac{1}{3}(x-q)(x-p)^3\right]_q^p - \int_q^p \frac{1}{3}(x-p)^3\,dx$

$= 0 + -\left[\frac{1}{12}(x-p)^4\right]_q^p$

$= -\frac{1}{12}\left(0-(q-p)^4\right)$

$= \frac{1}{12}(p-q)^4$ as $(q-p)^4 = (-(p-q))^4 = (p-q)^4$ ✓

</td><td>2</td></tr>
<tr><td></td><td>(ii)</td><td>

To find the area of AMNB we need the y-value at C to give the height and know the base is $AB = p-q$.

C is the turning point

$\frac{dy}{dx} = (x-q)\cdot 2(x-p) + (x-p)^2\cdot 1$

$= (x-p)(3x-2q-p)$

For max/min, $\frac{dy}{dx} = 0$

$\therefore (x-p)(3x-2q-p) = 0$

$x = p$ or $x = \frac{p+2q}{3}$ ✓

We know the turning point at B is when $x = p$

$\therefore$ at C, $x = \frac{p+2q}{3}$

$\therefore y = \left(\frac{p+2q}{3} - q\right)\left(\frac{p+2q}{3} - p\right)^2$

$= \left(\frac{p+2q-3q}{3}\right)\left(\frac{p+2q-3p}{3}\right)^2$

$= \left(\frac{p-q}{3}\right)\left(\frac{2q-2p}{3}\right)^2$

$= \left(\frac{p-q}{3}\right)\left(\frac{-2(p-q)}{3}\right)^2$

$= \frac{4}{27}(p-q)^3$ ✓

$\therefore A = (p-q)\left(\frac{4}{27}(p-q)^3\right)$

$= \frac{4}{27}(p-q)^4$

$A_a : A_{AMNB} = \frac{1}{12}(p-q)^4 : \frac{4}{27}(p-q)^4$

$= \frac{1}{12} : \frac{4}{27}$

$= \frac{1}{4} : \frac{4}{9}$

$= 1 : 16/9$ ✓

</td><td>3</td></tr>
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Question 16 (15 marks) **Marks**

(a)

a) Let C have coordinates (x, y, z)

Given $|\vec{OA}| = |\vec{OC}| = 1$

& $\triangle AOC$ is equilateral $\Rightarrow \angle AOC = \pi/3$.

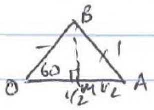
$\therefore \vec{OA} \cdot \vec{OC} = |\vec{OA}||\vec{OC}| \cos \pi/3$.

ie $\begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = 1 \times 1 \times \frac{1}{2}$.

$x + 0 + 0 = \frac{1}{2}$

$x = 1/2$ ✓

Now B is the altitude of $\triangle OAB$.



$MB = \sin 60°$

$= \sqrt{3}/2$.

$\therefore B = (\frac{1}{2}, \sqrt{3}/2, 0)$ ✓

So $\vec{OB} \cdot \vec{OC} = |\vec{OB}||\vec{OC}| \cos \pi/3$.

$\begin{pmatrix}\frac{1}{2}\\ \sqrt{3}/2\\0\end{pmatrix} \cdot \begin{pmatrix}\frac{1}{2}\\y\\z\end{pmatrix} = 1 \times 1 \times \frac{1}{2}$.

$\frac{1}{4} + \frac{\sqrt{3}}{2} y + 0 = \frac{1}{2}$.

$\frac{\sqrt{3}}{2} y = \frac{1}{4}$

$y = \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\sqrt{3}}{6}$ ✓

Finally $|\vec{OC}| = 1$

$\therefore \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{6})^2 + z^2} = 1$

$\frac{1}{4} + \frac{3}{36} + z^2 = 1$

$z^2 = 1 - \frac{1}{4} - \frac{1}{12}$

$= \frac{2}{3}$.

$z = \sqrt{2/3} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\sqrt{6}}{3}$. ✓

$\therefore C = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}\right)$

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(b) (i)

for $a, b \in \mathbb{R}^+$ *we have*

$\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 \geq 0$

$\frac{a}{b} - 2\sqrt{\frac{a}{b} \cdot \frac{b}{a}} + \frac{b}{a} \geq 0$

$\frac{a}{b} - 2 + \frac{b}{a} \geq 0$

$\frac{a}{b} + \frac{b}{a} \geq 2$ ✓

as required.

OR

$(a-b)^2 \geq 0$

$a^2 + b^2 \geq 2ab$

$\frac{a^2 + b^2}{ab} \geq 2 \quad$ *where* $(ab \geq 0)$

$\frac{a^2}{ab} + \frac{b^2}{ab} \geq 2$

$\frac{a}{b} + \frac{b}{a} \geq 2$

1

(ii)

Prove

$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$

From (i) we have:

$\frac{a+b}{b+c} + \frac{b+c}{a+b} \geq 2 \quad (\star)$ ✓

So

$\frac{a+b}{b+c} + \frac{b+c}{a+b} + \frac{b+c}{c+a} + \frac{c+a}{b+c} + \frac{c+a}{a+b} + \frac{a+b}{c+a}$

$\geq 2 + 2 + 2$ ✓

from summing the 3 variation of $(\star)$

Rearranging:

$\frac{a+b}{b+c} + \frac{c+a}{b+c} + \frac{b+c}{c+a} + \frac{a+b}{c+a} + \frac{b+c}{a+b} + \frac{c+a}{a+b} \geq 6$

$\frac{2a}{b+c} + \frac{b+c}{b+c} + \frac{2b}{c+a} + \frac{c+a}{c+a} + \frac{2c}{a+b} + \frac{a+b}{a+b} \geq 6$ ✓

$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) + 3 \geq 6$

$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 3/2$

as required.

3

(c)	(i)	$F = mg - kv$ $\therefore\ ma = mg - kv$ or $a = g - kv$ since $m = 1$ kg. $\therefore\ \frac{dv}{dt} = g - kv$ ✓ Integrating both sides w.r.t. time: $\int_0^v \frac{dv}{g-kv} = \int_0^t dt$ $\left[-\frac{1}{k}\ln\|g-kv\|\right]_0^v = [t]_0^t$ $-\frac{1}{k}(\ln(g-kv) - \ln g) = t$ $\ln\frac{g-kv}{g} = -kt$ $\frac{g-kv}{g} = e^{-kt}$ $g - kv = ge^{-kt}$ $kv = g - ge^{-kt}$ $v = \frac{g}{k}(1-e^{-kt})$ ✓ as required.	2
	(ii)	Again, we have $a = g - kv$ or $\frac{dv}{dt} = g - kv$ Integrating w.r.t time with adjusted limits we have: $\int_u^v \frac{dv}{g-kv} = \int_2^t dt$ ✓ $\left[-\frac{1}{k}\ln\|g-kv\|\right]_u^v = [t]_2^t$ $-\frac{1}{k}(\ln(g-kv) - \ln\|g-ku\|) = t-2$ $\ln\left(\frac{g-kv}{g-ku}\right) = -k(t-2)$ $\frac{g-kv}{g-ku} = e^{-k(t-2)}$ $g - kv = (g-ku)e^{-k(t-2)}$ $kv = g - (g-ku)e^{-k(t-2)}$ $v = \frac{g}{k} - e^{-k(t-2)}\left(\frac{g-ku}{k}\right)$ ✓ as required.	2

	(iii)	Sub $u = g/k$ in the expression for V in (ii) $V = \frac{g}{k} - e^{-k(t-2)}\left(\frac{g - k(g/k)}{k}\right)$ $= \frac{g}{k} - e^{-k(t-2)}\left(\frac{g-g}{k}\right)$ $= \frac{g}{k}$ ✓ $\therefore \frac{dy_2}{dt} = \frac{g}{k}$ $\int_0^{y_2} dy = \int_2^t \frac{g}{k}\,dt$ ✓ $y_2 = \frac{g}{k}(t-2)$	2
	(iv)	for collision, $y_1 = y_2$ $\therefore \frac{g}{k}\left(t + \frac{1}{k}(e^{-kt} - 1)\right) = \frac{g}{k}(t-2)$ $t + \frac{1}{k}(e^{-kt} - 1) = t - 2$ $e^{-kt} - 1 = -2k$ $e^{-kt} = 1 - 2k$ $-kt = \ln\|1-2k\|$ ✓ $t = -\frac{1}{k}\ln\|1-2k\|$ as required.	1